Name: ....

Fort St High School

2020 Trial HSC examination

Mathematics Extension 2

General Instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided at the back of this paper
- In Questions 11–16, show relevant mathematical reasoning and/ or calculations

Total Marks

100

Section I – 10 marks

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Detach the *Multiple-choice answer sheet* from the last page of this question booklet.

Section II – 90 marks

- Attempt Questions 11–16
- Allow about 2 hours and 45 minutes for this section

Section I
10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet (*provided on the last page of the booklet*) for Questions 1–10.

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1 Which expression is equal to $\int x^2 \sin\left(x^3\right)\,dx$?

A $\dfrac{x^3}{3}\sin\left(x^3\right)-\dfrac{1}{3}\int x^3 \sin\left(x^3\right)\,dx\ + C$

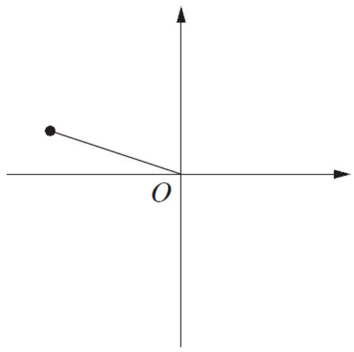
B $\dfrac{x^3}{3}\sin\left(x^3\right)-\dfrac{1}{3}\int x^3 \cos\left(x^3\right)\,dx\ + C$

C $\dfrac{1}{3}\cos\left(x^3\right)\ + C$

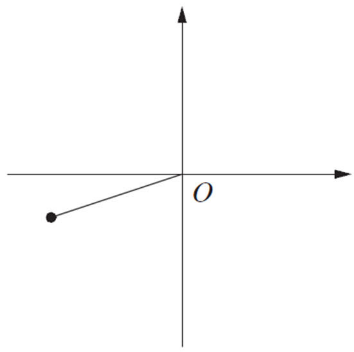
D $-\dfrac{1}{3}\cos\left(x^3\right)\ + C$

2 If $z = 2 - i$, which graph best represents $w = -\overline{z} + 2i$?

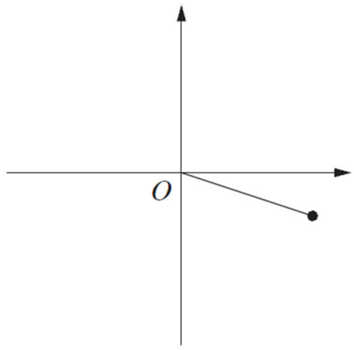
A



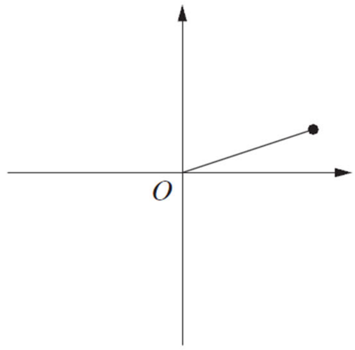
B



C



D



3 Which of the following is a primitive of $x\cos x$?

A $x\sin x + \sin x$

B $x\sin x + \cos x$

C $x\sin x - \cos x$

D $x\sin x - \sin x$

4 What is the magnitude of vector $\underset{\sim}{u} = 3\underset{\sim}{i} + 6\underset{\sim}{j} - 4\underset{\sim}{k}$?

A $\sqrt{61}$

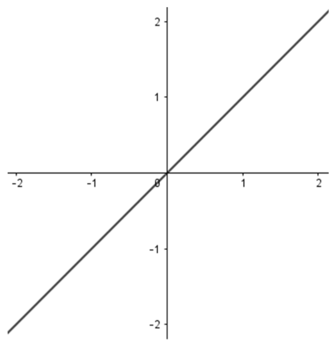
B $\sqrt{13}$

C $\sqrt{63}$

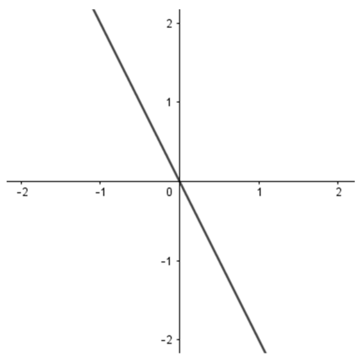
D $\sqrt{29}$

5 Which of the following diagrams best represents the solutions to the equation $|z-2| = |z-2i|$?

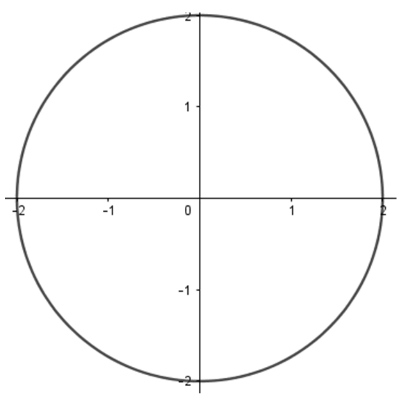
A



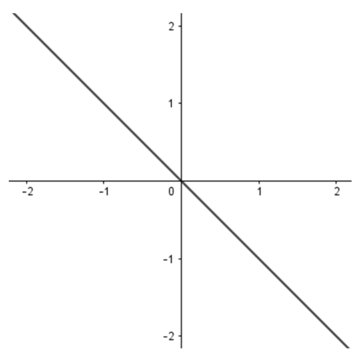
B



C



D



6 A constant force of magnitude F newtons accelerates a particle of mass 10 kg in a straight line from a speed of $6\ \text{ms}^{-1}$ to a speed of $20\ \text{ms}^{-1}$ over a distance of 8 m.

If there is no resistance, find the magnitude of F.

A $17.5\ \text{N}$

B $175\ \text{N}$

C $22\cdot 75\ \text{N}$

D $227\cdot 5\ \text{N}$

7 Simplify $\dfrac{e^{-\left(\frac{7i\pi}{6}\right)}}{e^{i\pi}}$

A $-\dfrac{\sqrt{3}}{2}+\dfrac{1}{2}i$

B $\dfrac{\sqrt{3}}{2}-\dfrac{1}{2}i$

C $\dfrac{1}{2}+\dfrac{\sqrt{3}}{2}i$

D $\dfrac{1}{2}-\dfrac{\sqrt{3}}{2}i$

8 The angle vector $\underset{\sim}{u}=\begin{bmatrix}2\\5\\4\end{bmatrix}$ makes with the x axis is closest to?

A 30°

B 73°

C 82°

D 87°

9 A particle is projected at an angle of α from the horizontal in a medium where resistance is proportional to velocity squared.

The horizontal component for acceleration of the particle is given by

A $\ddot{x} - kv$

B $\ddot{x} = -kv^2$

C $\ddot{x} = -kv\dot{x}$

D $\ddot{x} = -k\dot{x}^2$

10 A particle is moving along a straight line. Initially its displacement is at $x = 1$, its velocity is $v = 2$ and its acceleration is $a = 4$.

Which equation could describe the motion of the particle?

A $v = \sin(x-1) + 2$

B $v = 2e^{x-1}$

C $v^2 = x^2 + x + 2$

D $v = 2 + \ln x$

Section II

90 marks

Attempt Questions 11–16

Allow about 2 hours and 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available.
For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a new writing booklet.

(a) Let $z = 1 - \sqrt{3}i$, find complex numbers for

(i) $z\bar{z}$ **2**

(ii) z^2 **2**

(b) (i) Using Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$ find a similar result for $e^{-i\theta}$ **1**

(ii) Hence show that $\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$ **1**

(iii) Use the result from part (ii) to show $\cos^3\theta = \frac{1}{4}\cos 3\theta + \frac{3}{4}\cos\theta$ **3**

(c) (i) Find real numbers A and B such that **2**

$$\frac{3x^2 - x + 12}{(x-2)(x^2+2x+3)} \equiv \frac{A}{x-2} + \frac{Bx-3}{x^2+2x+3}.$$

(ii) Hence find $\int \frac{3x^2 - x + 12}{(x-2)(x^2+2x+3)}\, dx$. **4**

Question 12 (15 marks) Use a separate writing booklet.

(a) Let $z^4 = -2\sqrt{3} + 2i$.

(i) Write z^4 in modulus and argument form. **2**

(ii) Hence solve $z^4 = -2\sqrt{3} + 2i$ giving your answer in modulus-argument form. **3**

(b) Sketch the graph of the complex number z defined by $\arg\left(\dfrac{z-2}{z+2i}\right) = \dfrac{\pi}{2}$. **3**

(c) A rocket is fired vertically from the Earth's surface with initial speed V ms^{-1}. Assuming negligible air resistance, the acceleration experienced by the rocket is inversely proportional to the square of the distance from the centre of the earth and is directed towards the centre of the Earth.

(i) Show that the an expression for the velocity, v, of the rocket is given by $v^2 = \dfrac{2k}{x} + V^2 - \dfrac{2k}{R}$

where, k is the constant of proportionality and R is the distance from the Earth's centre to the surface. **3**

(ii) Show that at the Earth's surface $v = \sqrt{\dfrac{2gR^2}{x} + V^2 - 2gR}$,

where g is the acceleration due to gravity. **1**

(iii) Letting $g = 9 \cdot 8$ ms^{-2} and the distance to the Earth's surface R be 6400 km. Determine whether a rocket fired at 12 km/s is fast enough to escape Earth's gravitational pull.

Justify your answer. **3**

Question 13 (15 marks) Use a separate writing booklet.

(a) By completing the square, find $\int \frac{dx}{x^2+6x+13}$. **2**

(b) (i) What is the vector equation of the line through $P=(1,7,5)$ and parallel to $\underset{\sim}{i}-2\underset{\sim}{j}+2\underset{\sim}{k}$? **1**

(ii) What is the vector equation of the line through the points $A=(2,1,3)$ and $B=(2,-3,-1)$? **1**

(ii) Show that the lines found in parts (i) and (ii) are perpendicular. **1**

(iv) Find the point of intersection the lines found in parts (i) and (ii). **3**

(c) A swimmer in a pool stops swimming and is slowed with a resistive force $R=-mk\left(v_0+v^2\right)$, where m is the mass of the swimmer, v_0 is the velocity of the swimmer when she stops swimming, x is the distance, t is the time and v is the velocity of the swimmer after she stops swimming.

(i) Show that the distance before the swimmer comes to rest is $x=\frac{1}{2k}\ln\left(1+v_0\right)$ **3**

(ii) Show that the time t after she stops swimming is given by $t=\frac{1}{k\sqrt{v_0}}\tan^{-1}\left(\frac{v_0-v}{(1+v)\sqrt{v_0}}\right)$

and hence or otherwise find an expression for the time when the swimmer comes to rest. **4**

Question 14. (15 marks) Use a separate writing booklet.

(a) Evaluate $\int_{-1}^{\sqrt{3}} \frac{x^2}{\sqrt{4-x^2}}\,dx$ **4**

(b) Let ω be a cube root of unity where $\omega \neq 1$.

(i) Show that $1+\omega+\omega^2=0$. **1**

(ii) Consider the polynomial $\psi_n(x)=x^{4n+1}+(x+1)^{2n-1}$, where n is a positive integer.
Show that $\psi_n(\omega)=0$ and that $\psi_n(\omega^2)=0$. **4**

(iii) Hence, show that x^2+x+1 is a factor of $\psi_n(x)=x^{4n+1}+(x+1)^{2n-1}$. **2**

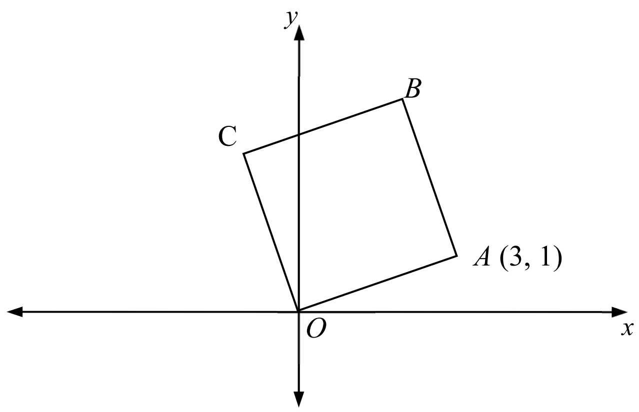
(c) The rise and fall of tides can be approximated to simple harmonic motion. **4**
At 11.00 a.m. the depth of water in a tidal lagoon is lowest at 2 m. The following high tide is at 5.21 p.m. with a depth of 6 m.

Calculate between what times a yacht could safely cross the lagoon if a minimum depth of water of 3 m is needed.

Question 15. (15 marks) Use a separate writing booklet.

(a) $OABC$ is a square in the Argand diagram. The point A represents the complex number $z = 3 + i$.

Find the complex numbers represented by B and C. **3**



(b) (i) Show that $\int_0^{\frac{\pi}{2}} (\sin x)^{2k} \cos x \, dx = \frac{1}{2k+1}$, where k is positive integer **2**

(ii) By writing $(\cos x)^{2n} = (1 - \sin^2 x)^n$, show that

$$\int_0^{\frac{\pi}{2}} (\cos x)^{2n+1} \, dx = \sum_{k=0}^{n} \frac{(-1)^k}{2k+1} \binom{n}{k}.$$ **4**

(iii) Hence, or otherwise, evaluate $\int_0^{\frac{\pi}{2}} (\cos x)^5 \, dx$ **2**

(c) A vector equation is given by $\underset{\sim}{v} = \underset{\sim}{c} + \underset{\sim}{a} \cos t + \underset{\sim}{b} \sin t$ where $\underset{\sim}{a} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}, \underset{\sim}{b} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$ and $\underset{\sim}{c} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$. **4**

Find the Cartesian equation of the vector and describe it geometrically.

Question 16. (15 marks) Use a separate writing booklet.

(a) Given $I_n = \int_0^1 \frac{1}{\left(1+x^4\right)^n} dx$, show that $I_{n+1} = \frac{4n-1}{4n} I_n + \frac{1}{n2^{n+2}}$ **3**

(b) Frank decides to go bungy-jumping. Frank falls, from rest, off a bridge attached by an elastic cable of length l metres. After Frank falls l metres he is slowed by the cable which exerts a force of $mgk\left(x-l\right)$, where m is his mass in kilograms, g is the constant acceleration due to gravity in ms^{-2} , k is the constant of proportionality. Let x m be the distance Frank has fallen and let v ms^{-1} be his speed at x.

There is negligible air resistance for the duration of Frank's jump.

(i) Show that $v^2 = 2gx - kg\left(x-l\right)^2$ for $x > l$. **4**

(ii) Show that Franks fall is first halted at $x = l + \frac{1}{k} + \sqrt{\frac{2l}{k} + \frac{1}{k^2}}$ **3**

(iii) Show that if the length of rope is 40 m and the constant of proportionality k is $\frac{1}{10}$ then the bridge needs to be at least 80 metres above the ground. **1**

(c) From a horizontal plane the path of particle projected at an angle α is given by the vector

$$\underset{\sim}{r}(t) = Vt\cos\alpha \underset{\sim}{i} + \left(Vt\sin\alpha - \frac{gt^2}{2}\right)\underset{\sim}{j}$$

Where V in ms^{-1} is the initial projection speed, t is the time in seconds and g is the acceleration due to gravity.

(i) Show that the Cartesian path of the particle is given by $y = x\tan\alpha - \frac{gx^2\sec^2\alpha}{2V^2}$ **2**

(ii) A vertical wall is a distance d metres from the origin.

Show that if $d < \frac{V^2}{g}$ the particle will strike the wall provided that $\beta < \alpha < \frac{\pi}{2} - \beta$, where $\beta = \frac{1}{2}\sin^{-1}\left(\frac{gd}{V^2}\right)$ **2**

End of examination.

Fort St High School

2020 **Trial HSC examination**

Mathematics Extension 2

Solutions

2020 Trial HSC Examination

Section 1 Solutions

1.	A ○	B ○	C ○	D ●
2.	A ●	B ○	C ○	D ○
3.	A ○	B ●	C ○	D ○
4.	A ●	B ○	C ○	D ○
5.	A ●	B ○	C ○	D ○
6.	A ○	B ○	C ○	D ●
7.	A ○	B ●	C ○	D ○
8.	A ○	B ●	C ○	D ○
9.	A ○	B ○	C ●	D ○
10.	A ○	B ●	C ○	D ○

Section I
10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet (*provided on the last page of the booklet*) for Questions 1–10.

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1 Which expression is equal to $\int x^2 \sin\left(x^3\right) dx$?

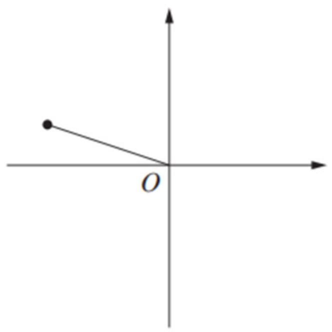
D $-\frac{1}{3}\cos\left(x^3\right) + C$

Working

$$\int x^2 \sin\left(x^3\right) dx = \frac{1}{3}\int 3x^2 \times \sin\left(x^3\right) dx$$
$$= -\cos x^3 + c$$

2 If $z = 2 - i$, which graph best represents $w = -\overline{z} + 2i$?

A



Working

$z = 2 - i$
$\overline{z} = 2 + i$

$w = -\left(2 + i\right) + 2i$
$= -2 + i$

3 Which of the following is a primitive of $x\cos x$?

B $x\sin x+\cos x$

Working

$$\begin{aligned}F(x)&=x\sin x-\int 1.\sin x\ dx\\&=x\sin x+\cos x\end{aligned}$$

4 What is the magnitude of vector $\underset{\sim}{u}=3\underset{\sim}{i}+6\underset{\sim}{j}-4\underset{\sim}{k}$?

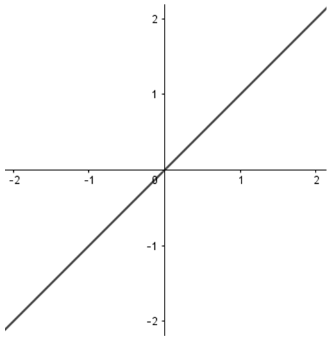
A $\sqrt{61}$

Working

$$\begin{aligned}|\underset{\sim}{u}|&=\sqrt{3^2+6^2+(-4)^2}\\&=\sqrt{61}\end{aligned}$$

5 Which of the following diagrams best represents the solutions to the equation $|z-2|=|z-2i|$?

A



Working

Let $z=x+iy$

$$\begin{aligned}|x+iy-2|&=|x+iy-2i|\\\sqrt{(x-2)^2+y^2}&=\sqrt{x^2+(y-2)^2}\\x^2-4x+4+y^2&=x^2+y^2-4y+4\\y&=x\end{aligned}$$

6 A constant force of magnitude *F* newtons accelerates a particle of mass 10 kg in a straight line from a speed of $6\ \text{ms}^{-1}$ to a speed of $20\ \text{ms}^{-1}$ over a distance of 8 m.

If there is no resistance, find the magnitude of *F*.

D $227\cdot 5\ \text{N}$

Working

$F = ma$ **or** from Physics

Since acceleration is constant

$$a = \frac{\Delta \frac{1}{2} v^2}{\Delta x}$$
$$= \frac{1}{2} \times \frac{20^2 - 6^2}{8 - 0}$$
$$= 22 \cdot 75$$

$$v^2 = u^2 - 2as$$
$$20^2 = 6^2 - 2a(8)$$
$$a = 22 \cdot 75$$

$$F = 10 \times 22 \cdot 75$$
$$= 227 \cdot 5\ N$$

7 Simplify $\dfrac{e^{-\left(\frac{7i\pi}{6}\right)}}{e^{i\pi}}$

B $\dfrac{\sqrt{3}}{2} - \dfrac{1}{2}i$

Working

$$\frac{e^{-\left(\frac{7i\pi}{6}\right)}}{e^{i\pi}} = e^{-\frac{7i\pi}{6} - i\pi}$$
$$= e^{-\frac{13\pi}{6}i}$$
$$= \cos\left(-\frac{13\pi}{6}\right) + i\sin\left(-\frac{13\pi}{6}\right)$$
$$= \cos\left(\frac{\pi}{6}\right) - i\sin\left(\frac{\pi}{6}\right)$$
$$= \frac{\sqrt{3}}{2} - i\frac{1}{2}$$

8 The angle vector $\underset{\sim}{u}=\begin{bmatrix}2\\5\\4\end{bmatrix}$ makes with the x axis is closest to?

B 73°

Working

Find the angle by finding the angle between the unit vector $\underset{\sim}{i}=\begin{bmatrix}1\\0\\0\end{bmatrix}$ and $\underset{\sim}{u}=\begin{bmatrix}2\\5\\4\end{bmatrix}$.

$$\begin{aligned}\cos\theta &= \frac{\underset{\sim}{u}\cdot\underset{\sim}{i}}{|\underset{\sim}{u}||\underset{\sim}{i}|}\\ &= \frac{2\times1+5\times0+4\times0}{\sqrt{2^2+5^2+4^2}\times1}\\ &= \frac{2}{\sqrt{45}}\\ &\approx 73^\circ\end{aligned}$$

9 A particle is projected at an angle of α from the horizontal in a medium where resistance is proportional to velocity squared.

The horizontal component for acceleration of the particle is given by

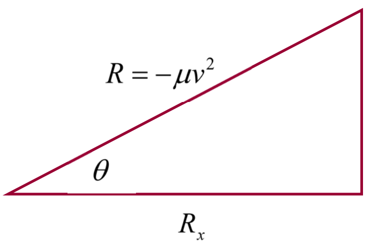
C $\ddot{x}=-kv\dot{x}$

Working

In projected motion resistance is opposite to direction.

From the resistance diagram

$$\begin{aligned}R_x &= -\mu v^2\cos\\ m\ddot{x} &= -\mu v(v\cos\theta) \qquad \text{as } \dot{x}=v\cos\theta\\ &= -\frac{\mu}{m}v\dot{x}\\ &= -kv\dot{x}\end{aligned}$$



10 A particle is moving along a straight line. Initially its displacement is at $x=1$, its velocity is $v=2$ and its acceleration is $a=4$.

Which equation could describe the motion of the particle?

B $v=2e^{x-1}$

Working

Finding the equation that gives the correct value for *a*

A:

$$\frac{1}{2}v^2=\frac{1}{2}\left(\sin(x-1)+2\right)^2$$

$$\begin{aligned}\frac{d\,\frac{1}{2}v^2}{dx}&=\left(\sin(x-1)+2\right)\cos(x-1)\\ a&=\left(\sin(1-1)+2\right)\cos(1-1)\\ &=\left(\sin 0+2\right)\cos 0\\ &\neq 4\end{aligned}$$

B:

$$\begin{aligned}v&=2e^{x-1}\\ \frac{1}{2}v^2&=2e^{2x-2}\end{aligned}$$

$$\begin{aligned}\frac{d\,\frac{1}{2}v^2}{dx}&=4e^{2x-2}\\ a&=4e^0\\ &=4\end{aligned}$$

C:

$$\frac{1}{2}v^2=\frac{1}{2}\left(x^2+x+2\right)^2$$

$$\begin{aligned}\frac{d\,\frac{1}{2}v^2}{dx}&=\left(x^2+x+2\right)(2x+1)\\ a&=\left(1^2+1+2\right)(2+1)\\ &\neq 4\end{aligned}$$

D:

$$\frac{1}{2}v^2=\frac{1}{2}(2+\ln x)^2$$

$$\begin{aligned}\frac{d\,\frac{1}{2}v^2}{dx}&=(2+\ln x)\left(\frac{1}{x}\right)\\ a&=(2)(1)\\ &\neq 4\end{aligned}$$

Section II

Question 11 (15 marks)

(a) Let $z = 1-\sqrt{3}i$, find

(i) $z\bar{z}$

Solution

$$z\bar{z} = \left(1-\sqrt{3}i\right)\left(1+\sqrt{3}i\right)$$
$$= 1-3i^2$$
$$= 4$$

Mark	Guideline	Marker's comment
2	Correct response	Well answered with mistakes tending to be careless arithmetic
1	One error	

(ii) z^2

Solution

$$z^2 = \left(1-\sqrt{3}i\right)^2$$
$$= 1-2\sqrt{3}i+3i^2$$
$$= -2-2\sqrt{3}i$$

Mark	Guideline	Marker's comment
2	Correct response	Well answered with mistakes tending to be careless arithmetic
1	One error	

(b) (i) Using Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$ find a similar result for $e^{-i\theta}$

Solution

$$e^{-i\theta} = \cos\theta - i\sin\theta$$

Mark	Guideline	Marker's comment
1	Correct response	Well done.

(ii) Hence show that $\cos\theta = \frac{1}{2}\left(e^{i\theta}+e^{-i\theta}\right)$

$$e^{i\theta}+e^{-i\theta} = \cos\theta + i\sin\theta + \cos\theta - i\sin\theta$$
$$2\cos\theta = e^{i\theta}+e^{-i\theta}$$
$$\cos\theta = \frac{1}{2}\left(e^{i\theta}+e^{-i\theta}\right)$$

Mark	Guideline	Marker's comment
1	Correct response	Students should be mindful that *show* questions require explicit and thorough steps.

(iii) Use the result from part (ii) to show $\cos^3\theta = \frac{1}{4}\cos 3\theta + \frac{3}{4}\cos\theta$

Solution

$$
\begin{aligned}
LHS &= \cos^3\theta \\
&= \left[\frac{1}{2}\left(e^{i\theta}+e^{-i\theta}\right)\right]^3 \\
&= \frac{1}{8}\left[\left(e^{i\theta}\right)^3+3\left(e^{i\theta}\right)^2\left(e^{-i\theta}\right)+3\left(e^{i\theta}\right)\left(e^{-i\theta}\right)^2+\left(e^{-i\theta}\right)^3\right] \\
&= \frac{1}{8}\left[e^{3i\theta}+e^{-3i\theta}+3e^{i\theta}+3e^{-i\theta}\right] \\
&= \frac{1}{4}\left(\frac{1}{2}\left(e^{3i\theta}+e^{-3i\theta}\right)\right)+\frac{3}{4}\left(\frac{1}{2}\left(e^{i\theta}+e^{-i\theta}\right)\right) \\
&= \frac{1}{4}\cos 3\theta+\frac{3}{4}\cos\theta \\
&= RHS
\end{aligned}
$$

Mark	Guideline	Marker's comment
3	Correct response	Some students did not use the previous result. Many students could not use the binomial expansion on the previous result.
2	One error	
1	For partial solution	

(c) (i) Find real numbers A and B such that $\dfrac{3x^2-x+12}{(x-2)(x^2+2x+3)} \equiv \dfrac{A}{x-2}+\dfrac{Bx-3}{x^2+2x+3}$

Solution

$$\frac{3x^2-x+12}{(x-2)(x^2+2x+3)} \equiv \frac{A}{x-2}+\frac{Bx-3}{x^2+2x+3}$$

$$3x^2-x+12 = A\left(x^2+2x+3\right)+\left(Bx-3\right)\left(x-2\right)$$

Let $x=2$

$$22 = 11A$$

$$A = 2$$

$$3x^2-x+12 = 2x^2+4x+6+Bx^2-2Bx-3x+6$$

$$3x^2-x+12 = \left(B+2\right)x^2+\left(1-2B\right)x+12$$

Comparing coefficients

$$3 = B+2$$

$$B = 1$$

Mark	Guideline	Marker's comment
2	Correct response	Generally well answered.
1	One correct response	

(ii) Hence find $\displaystyle\int \frac{3x^2 - x + 12}{(x-2)(x^2+2x+3)}\, dx$.

Solution

$$\begin{aligned}
\int \frac{3x^2 - x + 12}{(x-2)(x^2+2x+3)}\, dx &= \int \left(\frac{2}{x-2} + \frac{x-3}{x^2+2x+3} \right) dx \\
&= \int \left(\frac{2}{x-2} + \frac{x+1}{x^2+2x+3} - \frac{4}{x^2+2x+3} \right) dx \\
&= 2\ln|x-2| + \frac{1}{2}\ln\left|x^2+2x+3\right| - \int \frac{4}{(x+1)^2+2} dx \\
&= 2\ln|x-2| + \frac{1}{2}\ln\left|x^2+2x+3\right| - 2\sqrt{2}\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right) + C
\end{aligned}$$

Mark	Guideline	Marker's comment
4	Correct response	Generally well answered with any mistakes were with the multiple of $\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right)$ Don't forget to write + C
3	One error or omitting constant	
2	For two errors	
1	Obtaining $2\ln\|x-2\|$ as part of the solution	

Question 12 (15 marks) Use a separate writing booklet.

(a) Let $z^4 = -2\sqrt{3} + 2i$. (i) Write z^4 in modulus and argument form.

Solution

$$\left|z^4\right| = \sqrt{\left(-2\sqrt{3}\right)^2 + 2^2} = 4$$

$$\arg z^4 = \tan^{-1}\frac{2}{-2\sqrt{3}} = \frac{5\pi}{6}$$

Therefore $z^4 = 4\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$

Mark	Guideline	Marker's comment
2	Correct response	
1	One correct response	

(ii) Hence solve $z^4 = -2\sqrt{3} + 2i$ giving your answer in modulus-argument form.

Solution

Let $z = |z| cis\theta$

$$\left|z^4\right| = 4 \qquad |z| = \sqrt{2}$$

$$\arg z^4 = \frac{5\pi}{6}$$

$$4\arg z = \frac{5\pi}{6} + 2k\pi$$

$$\arg z = \frac{(12k+5)\pi}{24}$$

When $k = 0$ $\quad \theta = \frac{5\pi}{24}$

$k = 1$ $\quad \theta = \frac{17\pi}{24}$

$k = -1$ $\quad \theta = -\frac{7\pi}{24}$

$k = -2$ $\quad \theta = -\frac{19\pi}{24}$

Therefore

$$z_1 = \sqrt{2}\left(\cos\frac{5\pi}{24} + i\sin\frac{5\pi}{24}\right)$$

$$z_2 = \sqrt{2}\left(\cos\frac{17\pi}{24} + i\sin\frac{17\pi}{24}\right)$$

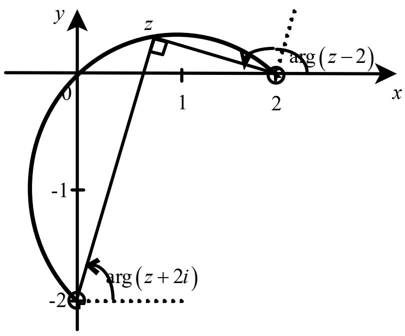
$$z_3 = \sqrt{2}\left(\cos\frac{7\pi}{24} - i\sin\frac{7\pi}{24}\right)$$

$$z_4 = \sqrt{2}\left(\cos\frac{17\pi}{24} - i\sin\frac{17\pi}{24}\right)$$

Mark	Guideline	Marker's comment
3	Correct response	
2	Principal arguments not given	
1	For finding $\lvert z \rvert$	

(b) Sketch the graph of the complex number z defined by $\arg\left(\dfrac{z-2}{z+2i}\right)=\dfrac{\pi}{2}$.

Solution



Mark	Guideline	Marker's comment
3	Correct response	
2	For arc with incorrect values (points)	
1	For arc on wrong side or circle	

(c) A rocket is fired vertically from the Earth's surface with initial speed V ms^{-1}. Assuming negligible air resistance, the acceleration experienced by the rocket is inversely proportional to the square of the distance from the centre of the earth and is directed towards the centre of the Earth.

(i) Show that the an expression for the velocity, v, of the rocket is given by $v^2=\dfrac{2k}{x}+V^2-\dfrac{2k}{R}$ where, k is the constant of proportionality and R is the distance from the Earth's centre to the surface.

Solution

$$\ddot{x}=-\frac{k}{x^2}$$

$$\frac{1}{2}\frac{dv^2}{dx}=-\frac{k}{x^2}$$

$$\frac{1}{2}v^2=\frac{k}{x}+C$$

When $x=R, v=V$ $\quad\Rightarrow C=\dfrac{1}{2V^2}-\dfrac{k}{R}$

$$\frac{1}{2}v^2=\frac{k}{x}+\frac{1}{2V^2}-\frac{k}{R}$$

$$v^2=\frac{2k}{x}+V^2-\frac{2k}{R}$$

Mark	Guideline	Marker's comment
3	Correct response	
2	For one error in working	
1	For stating $\ddot{x}=-\dfrac{k}{x^2}$	

(ii) Show that at the Earth's surface $v=\sqrt{\dfrac{2gR^2}{x}+V^2-2gR}$,

where g is the acceleration due to gravity.

Solution

At the Earth's surface
$$\ddot{x}=-\frac{k}{x^2}$$
$$-g=-\frac{k}{R^2}$$
$$k=gR^2$$

$$v^2=\frac{2\left(gR^2\right)}{x}+V^2-\frac{2\left(gR^2\right)}{R}$$
$$v=\sqrt{\frac{2gR^2}{x}+V^2-2gR} \qquad as\ \ v>0$$

Mark	Guideline	Marker's comment
1	Correct response	

(iii) Letting $g=9\cdot 8\ \text{ms}^{-2}$ and the distance to the Earth's surface R be 6400 km. Determine whether a rocket fired at 12 km/s is fast enough to escape Earth's gravitational pull.

Justify your answer.

Solution

Now to escape Earth's pull the limiting value for v has to be positive.

$$as\ \ x\to\infty \qquad \frac{2gR^2}{x}\to 0 \qquad \Rightarrow \qquad \lim_{x\to\infty} v=\sqrt{V^2-2gR}$$

$$\sqrt{V^2-2gR}>0$$
$$V^2-2gR>0$$
$$V^2>2gR$$
$$>2\times\left(0\cdot 0098\right)\times 6400 \qquad \text{Note: conversion of units}$$
$$>125\cdot 44$$
$$V>11\cdot 2\ \text{kms}^{-1} \qquad \text{as} \qquad V>0$$

Therefore the rocket fired at $12\ \text{kms}^{-1}$ will be enough to Earth's gravitational pull as it greater than the required $11\cdot 2\ \text{kms}^{-1}$.

Mark	Guideline	Marker's comment
3	Correct response	
2	For finding a value V needs to exceed with conversion error	
1	For finding the limit for v	

Question 13 (15 marks) Use a separate writing booklet.

(a) By completing the square, find $\int \frac{dx}{x^2+6x+13}$.

Solution

$$\int \frac{dx}{x^2+6x+13} = \int \frac{dx}{x^2+6x+9+4}$$
$$= \int \frac{dx}{(x+3)^2+4}$$
$$= \frac{1}{2}\tan^{-1}\left(\frac{x+3}{2}\right)+C$$

Mark	Guideline	Marker's comment
2	Correct response	Well answered.
1	For completing the square	

(b) (i) What is the vector equation of the line through $P=(1,7,5)$ and parallel to $\underset{\sim}{i}-2\underset{\sim}{j}+2\underset{\sim}{k}$?

Solution

$$\underset{\sim}{r} = \begin{bmatrix}1\\7\\5\end{bmatrix} + \mu\begin{bmatrix}1\\-2\\2\end{bmatrix}$$

Mark	Guideline	Marker's comment
1	Correct response	Well answered.

(ii) What is the vector equation of the line through the points $A=(2,1,3)$ and $B=(2,-3,-1)$?

Solution

$$\underset{\sim}{s} = \begin{bmatrix}2\\1\\3\end{bmatrix} + \lambda\begin{bmatrix}2-2\\-3-1\\-1-3\end{bmatrix}$$
$$= \begin{bmatrix}2\\1\\3\end{bmatrix} + \lambda\begin{bmatrix}0\\-4\\-4\end{bmatrix}$$

Mark	Guideline	Marker's comment
1	Correct response	Well answered with multiple correct responses.

(ii) Show that the lines found in parts (i) and (ii) are perpendicular.

Solution

The lines are perpendicular if the dot product of their directional vectors is zero.

$$\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} \bullet \begin{bmatrix} 0 \\ -4 \\ -4 \end{bmatrix} = 1\times 0 + (-2)\times(-4) + 2\times(-4)$$

$$= 0 + 8 - 8$$

$$= 0$$

Therefore the lines are perpendicular

Mark	Guideline	Marker's comment
1	Correct response	Some students did not make the connection between perpendicular lines and dot product.

(iv) Find the point of intersection the lines found in parts (i) and (ii).

Solution

$$\begin{bmatrix} 1+\mu \\ 7-2\mu \\ 5+2\mu \end{bmatrix} = \begin{bmatrix} 2 \\ 1-4\lambda \\ 3-4\lambda \end{bmatrix}$$

$1+\mu = 2$

$\mu = 1$

$7 - 2(1) = 1 - 4\lambda$

$4\lambda = -4$

$\lambda = -1$

Check values satisfy third component $5 + 2(1) = 3 - 4(-1)$

Therefore the point of intersection is $(2,5,7)$

Mark	Guideline	Marker's comment
3	Correct response	Generally well answered.
2	For finding μ *or* λ	
1	For finding equating components	

(c) A swimmer in a pool stops swimming and is slowed with a resistive force $R = -mk\left(v_0 + v^2\right)$, where m is the mass of the swimmer, v_0 is the velocity of the swimmer when she stops swimming, x is the distance, t is the time and v is the velocity of the swimmer after she stops swimming.

(i) Show that the distance before the swimmer comes to rest is $x = \frac{1}{2k}\ln\left(1 + v_0\right)$

Solution

$$R = -mk\left(v_0 + v^2\right)$$

$$\ddot{x} = -k\left(v_0 + v^2\right)$$

$$\frac{1}{2}\frac{dv^2}{dx} = -k\left(v_0 + v^2\right)$$

$$-2k\frac{dx}{dv^2} = \frac{1}{v_0 + v^2}$$

$$-2kx = \ln\left|v_0 + v^2\right| + C$$

When $x = 0,\ v = v_0$ $\quad \Rightarrow C = -\ln\left|v_0 + {v_0}^2\right|$

$$-2kx = \ln\left|v_0 + v^2\right| - \ln\left|v_0 + {v_0}^2\right|$$

$$x = \frac{1}{2k}\ln\left|\frac{v_0 + {v_0}^2}{v_0 + v^2}\right|$$

Now when the swimmer comes to rest $v = 0$

$$x = \frac{1}{2k}\ln\left|\frac{v_0 + {v_0}^2}{v_0}\right|$$

$$= \frac{1}{2k}\ln\left|1 + v_0\right|$$

Mark	Guideline	Marker's comment
3	Correct response	Generally well answered. Some students using definite integrals did not apply the procedure to both sides of their equation and were therefore unable to show where the swimmer come to rest.
2	For finding $x = \frac{1}{2k}\ln\left\|\frac{v_0 + {v_0}^2}{v_0 + v^2}\right\|$	
1	For partial solution	

(ii) Show that the time t after she stops swimming is given by $t = \frac{1}{k\sqrt{v_0}} \tan^{-1}\left(\frac{v_0 - v}{(1+v)\sqrt{v_0}} \right)$ and hence or otherwise find an expression for the time when the swimmer comes to rest.

Solution

$$R = -mk\left(v_0 + v^2\right)$$

$$m\ddot{x} = -mk\left(v_0 + v^2\right)$$

$$\frac{dv}{dt} = -k\left(v_0 + v^2\right)$$

$$\frac{dt}{dv} = -\frac{1}{k\left(v_0 + v^2\right)}$$

$$-kt = \frac{1}{\left(\sqrt{v_0}\right)^2 + v^2}$$

$$-kt = \frac{1}{\sqrt{v_0}} \tan^{-1} \frac{v}{\sqrt{v_0}} + C$$

When $t = 0, v = v_0 \quad \Rightarrow \quad C = -\frac{1}{\sqrt{v_0}} \tan^{-1} \sqrt{v_0}$

$$-kt = \frac{1}{\sqrt{v_0}} \tan^{-1} \frac{v}{\sqrt{v_0}} - \frac{1}{\sqrt{v_0}} \tan^{-1} \sqrt{v_0}$$

$$t = \frac{1}{k\sqrt{v_0}} \left(\tan^{-1} \sqrt{v_0} - \tan^{-1} \frac{v}{\sqrt{v_0}} \right)$$

Now let $\alpha = \tan^{-1} \sqrt{v_0}$ and $\beta = \tan^{-1} \frac{v}{\sqrt{v_0}}$

$\tan\alpha = \sqrt{v_0}$ and $\tan\beta = \frac{v}{\sqrt{v_0}}$

Now

$$\tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$$

$$=\frac{\sqrt{v_0}-\dfrac{v}{\sqrt{v_0}}}{1+\sqrt{v_0}\times\dfrac{v}{\sqrt{v_0}}}$$

$$=\frac{\dfrac{v_0-v}{\sqrt{v_0}}}{1+v}$$

$$=\frac{v_0-v}{(1+v)\sqrt{v_0}}$$

$$\alpha-\beta=\tan^{-1}\left(\frac{v_0-v}{(1+v)\sqrt{v_0}}\right)$$

$$t=\frac{1}{k\sqrt{v_0}}\left(\tan^{-1}\sqrt{v_0}-\tan^{-1}\frac{v}{\sqrt{v_0}}\right)$$

$$=\frac{1}{k\sqrt{v_0}}\tan^{-1}\left(\frac{v_0-v}{(1+v)\sqrt{v_0}}\right)$$

The swimmer comes to rest when $v=0$

$$=\frac{1}{k\sqrt{v_0}}\tan^{-1}\left(\frac{v_0}{\sqrt{v_0}}\right)$$

$$=\frac{1}{k\sqrt{v_0}}\tan^{-1}\sqrt{v_0}$$

Mark	Guideline	Marker's comment
4	Correct response	Generally well answered. What troubled most students was simplifying the two inverse functions. $t=\frac{1}{k\sqrt{v_0}}\left(\tan^{-1}\sqrt{v_0}-\tan^{-1}\frac{v}{\sqrt{v_0}}\right)$ Students should be mindful to set their solution out in a logical manner and if required use more than one page; rather than 'snaking' their solution around the page.
3	One error	
2	For correctly evaluating C	
1	For finding $-kt=\frac{1}{\sqrt{v_0}}\tan^{-1}\frac{v}{\sqrt{v_0}}+C$	

Question 14. (15 marks) Use a separate writing booklet.

(a) Evaluate $\int_{-1}^{\sqrt{3}} \frac{x^2}{\sqrt{4-x^2}}\,dx$

Solution

Let $x = 2\sin\theta$ $\qquad x=\sqrt{3}\ \ \theta = \frac{\pi}{3}$

$\frac{dx}{d\theta} = 2\cos\theta$ $\qquad x=-1\ \ \theta = -\frac{\pi}{6}$

$dx = 2\cos\theta\ d\theta$

$$\begin{aligned}
\int_{-1}^{\sqrt{3}} \frac{x^2}{\sqrt{4-x^2}}\,dx &= \int_{-\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{(2\sin\theta)^2}{\sqrt{4-(2\sin\theta)^2}}.2\cos\theta\ d\theta \\
&= \int_{-\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{4\sin^2\theta}{\sqrt{4-(2\sin\theta)^2}}.2\cos\theta\ d\theta \\
&= \int_{-\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{8\sin^2\theta\cos\theta}{\sqrt{4(1-\sin^2\theta)}}\ d\theta \\
&= \int_{-\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{8\sin^2\theta\cos\theta}{2|\cos\theta|}\ d\theta \\
&= 4\int_{-\frac{\pi}{6}}^{\frac{\pi}{3}} \sin^2\theta\ d\theta \qquad \text{as } \cos\theta > 0 \text{ for } -\frac{\pi}{6} \le \theta \le \frac{\pi}{3} \\
&= 4\int_{-\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{2} - \frac{1}{2}\cos 2\theta\ d\theta \\
&= 2\int_{-\frac{\pi}{6}}^{\frac{\pi}{3}} 1 - \cos 2\theta\ d\theta \\
&= 2\left[\theta - \frac{1}{2}\sin 2\theta\right]_{-\frac{\pi}{6}}^{\frac{\pi}{3}} \\
&= 2\left[\left(\frac{\pi}{3} - \frac{1}{2}\sin\frac{2\pi}{3}\right) - \left(-\frac{\pi}{6} - \frac{1}{2}\sin\frac{-\pi}{3}\right)\right] \\
&= 2\left[\frac{\pi}{3} - \frac{\sqrt{3}}{4} + \frac{\pi}{6} - \frac{\sqrt{3}}{4}\right] \\
&= 2\left(\frac{\pi}{2} - \frac{\sqrt{3}}{2}\right) \\
&= \pi - \sqrt{3}
\end{aligned}$$

Mark	Guideline	Marker's comment
4	Correct response	Student's who could identify the correct substitution had success with this question apart from some minor errors. Students are reminded that limits of the integral must be changed when the variable of integral is changed.
3	One error	
2	For appropriate substitution	
1	Partial solution with errors	

(b) Let ω be a cube root of unity where $\omega \neq 1$.

(i) Show that $1+\omega+\omega^2=0$.

Solution

If ω is a cube root of unity then

$$\omega^3 = 1$$
$$\omega^3 - 1 = 0$$
$$(\omega-1)(\omega^2+\omega+1) = 0$$
$$\omega^2+\omega+1 = 0 \quad \text{since} \quad \omega \neq 1$$

Mark	Guideline	Marker's comment
1	Correct response	Generally Answered well. A number of students did not identify that $\omega \neq 1$ as part of their solution.

(ii) Consider the polynomial $\psi_n(x) = x^{4n+1} + (x+1)^{2n-1}$, where n is a positive integer.

Show that $\psi_n(\omega)=0$ and that $\psi_n(\omega^2)=0$.

Solution

$$\begin{aligned}
\psi_n(\omega) &= (\omega)^{4n+1} + (\omega+1)^{2n-1} \\
&= \omega^{4n+1} + (-\omega^2)^{2n-1} && \text{as} \quad 1+\omega=-\omega^2 \\
&= \omega^{4n+1} + (-1)^{2n-1}(\omega^2)^{2n-1} \\
&= \omega^{4n+1} - \omega^{4n-2} && \text{as} \quad (-1)^{2n-1} = -1 \\
&= \omega^{4n-2}(\omega^3-1) \\
&= \omega^{4n-2}(0) && \text{as cube root of unity } \omega^3-1=0 \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\psi_n(\omega^2) &= (\omega^2)^{4n+1} + (\omega^2+1)^{2n-1} \\
&= \omega^{8n+2} + (-\omega)^{2n-1} && \text{as} \quad 1+\omega=-\omega^2 \\
&= \omega^{6n+2n+2} + (-1)^{2n-1}(\omega)^{2n-1} \\
&= \omega^{6n} \times \omega^{2n+2} - \omega^{2n-1} \\
&= (\omega^3)^{2n} \times \omega^{2n+2} - \omega^{2n-1} && \text{as cube root of unity } \omega^3 = 1 \\
&= \omega^{2n+2} - \omega^{2n-1} \\
&= \omega^{2n-1}(\omega^3-1) \\
&= \omega^{2n-1}(0) \\
&= 0
\end{aligned}$$

Mark	Guideline	Marker's comment
4	Correct response	Students used various strategies to answer this question. The most common error was not identifying that $(-1)^{2n-1} = -1$ explicitly, resulting in loss of mark. As this was a show question, explicit setting out of reasoning was essential for achieving full marks.
3	One error with correct procedure	
2	One identity shown	
1	Substituted $1+\omega=-\omega^2$	

(iii) Hence, show that x^2+x+1 is a factor of $\psi_n(x)=x^{4n+1}+(x+1)^{2n-1}$. **2**

Solution

From part (ii) if $\psi_n(\omega)=0$ and $\psi_n(\omega^2)=0$ then $(x-\omega)$ and $(x-\omega^2)$ are factors of $\psi_n(x)=x^{4n+1}+(x+1)^{2n-1}$

Now this implies $(x-\omega)(x-\omega^2)$ is also a factor.

$$\begin{aligned}(x-\omega)(x-\omega^2)&=x^2-\omega^2x-\omega x+\omega^3\\&=x^2-(\omega+\omega^2)x+1\\&=x^2-(-1)x+1\\&=x^2+x+1\end{aligned}$$

Therefore x^2+x+1 is a factor of $\psi_n(x)=x^{4n+1}+(x+1)^{2n-1}$

Mark	Guideline	Marker's comment
2	Correct response	Students who identified that $(x-\omega)(x-\omega^2)$ was also a factor achieved a full solution.
1	For recognising $(x-\omega)(x-\omega^2)$ is also a factor.	

(c) The rise and fall of tides can be approximated to simple harmonic motion.
At 11.00 a.m. the depth of water in a tidal lagoon is lowest at 2 m. The following high tide is at 5.21 p.m. with a depth of 6 m.

Calculate between what times a yacht could safely cross the lagoon if a minimum depth of 3 m of water is needed.

Solution

Simple Harmonic Motion has the form $x=A\cos nt+B$ if starting at the end of motion. Let the time t start at low tide. The difference in depth between high and low tide is 4 m implies the amplitude is 2 and $A=-2$ since we're starting at low tide. The centre of motion would be at $x=4$.

The time it takes for the lagoon to return to low tide is 6 hrs and 21 min $\times 2=762$ mins.

$$T=\frac{2\pi}{n}$$
$$762=\frac{2\pi}{n}$$
$$n=\frac{\pi}{381}$$

Now the depth of water required is 3 m and solving for t

$$x=-2\cos\frac{\pi t}{381}+4$$
$$3=-2\cos\frac{\pi t}{381}+4$$
$$\frac{1}{2}=\cos\frac{\pi t}{381}$$
$$\frac{\pi t}{380}=2k\pi\pm\cos^{-1}\left(\frac{1}{2}\right)$$

Considering on the positive answers as $t > 0$

$$\frac{\pi t}{381} = \frac{\pi}{3}, \frac{5\pi}{3}, \ldots$$

$$t = 127, 635, \ldots \text{ mins}$$

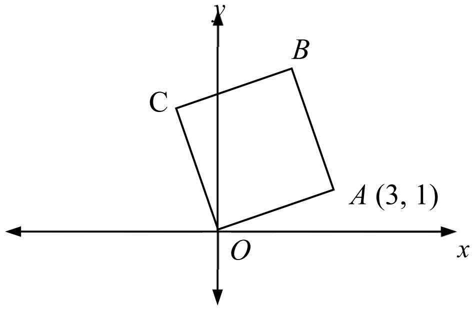
Therefore the time a yacht can safely cross the lagoon is $11:07$ am and $9:35$ pm.

Mark	Guideline	Marker's comment
4	Correct response	This question was answered well by most students. Students are encouraged to pause and read the question carefully as a common mistake was to set the period to the time from low tide to high tide, rather than a full cycle.
3	Solving for *t* without answering the question	
2	Correct displacement equation	
1	Identify format for displacement	

Question 15. (15 marks) Use a separate writing booklet.

(a) $OABC$ is a square in the Argand diagram. The point A represents the complex number $z = 3 + i$.

Find the complex numbers represented by B and C.



Solution

$$\begin{aligned} C &= \overrightarrow{OA} \text{ rotated by } 90^\circ \\ &= (3+i)i \\ &= -1+3i \end{aligned}$$

$$\begin{aligned} B &= \overrightarrow{OA} + \overrightarrow{OC} \\ &= 3+i-1+3i \\ &= 2+4i \end{aligned}$$

Mark	Guideline	Marker's comment
3	Correct response	Most students achieved full marks for this question.
2	For correct procedures with an error	
1	For demonstrating one concept	

(b) (i) Show that $\int_0^{\frac{\pi}{2}} (\sin x)^{2k} \cos x \, dx = \frac{1}{2k+1}$, where k is positive integer

Solution

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \cos x (\sin x)^{2k} \, dx &= \left[\frac{(\sin x)^{2k+1}}{2k+1} \right]_0^{\frac{\pi}{2}} \\ &= \left(\frac{\left(\sin \frac{\pi}{2}\right)^{2k+1}}{2k+1} \right) - \left(\frac{(\sin 0)^{2k+1}}{2k+1} \right) \\ &= \frac{1}{2k+1} \end{aligned}$$

Mark	Guideline	Marker's comment
2	Correct response	Answered well by most students.
1	For recognising $\int f'(x)[f(x)]^n \, dx = \frac{[f(x)]^n}{n+1}$	

(ii) By writing $(\cos x)^{2n} = (1-\sin^2 x)^n$, show that $\int_0^{\frac{\pi}{2}} (\cos x)^{2n+1}\, dx = \sum_{k=0}^{n} \frac{(-1)^k}{2k+1}\binom{n}{k}$.

Solution

$$\int_0^{\frac{\pi}{2}} (\cos x)^{2n+1}\, dx = \int_0^{\frac{\pi}{2}} \cos x (\cos x)^{2n}\, dx$$

$$= \int_0^{\frac{\pi}{2}} \cos x (\cos^2 x)^n\, dx$$

$$= \int_0^{\frac{\pi}{2}} \cos x (1-\sin^2 x)^n\, dx$$

$$= \int_0^{\frac{\pi}{2}} \cos x \left[\binom{n}{0}(-\sin^2 x)^0 + \binom{n}{1}(-\sin^2 x)^1 + \binom{n}{2}(-\sin^2 x)^2 + \ldots + \binom{n}{n}(-\sin^2 x)^n\right] dx$$

$$= \int_0^{\frac{\pi}{2}} \left[\binom{n}{0}\cos x(-\sin^2 x)^0 + \binom{n}{1}\cos x(-\sin^2 x)^1 + \ldots + \binom{n}{n}\cos x(-\sin^2 x)^n\right] dx$$

$$= \int_0^{\frac{\pi}{2}} \sum_{k=0}^{n} \binom{n}{k}(-1)^k \cos x (\sin^2 x)^k\, dx$$

$$= \sum_{k=0}^{n} \binom{n}{k}(-1)^k \int_0^{\frac{\pi}{2}} \cos x (\sin x)^{2k}\, dx$$

$$= \sum_{k=0}^{n} \binom{n}{k}\frac{(-1)^k}{2k+1}.$$

Mark	Guideline	Marker's comment
4	Correct response	Most student's struggled with this questions. Student's who identified the binomial expansion could not provide a clear solution, loosing marks for poor and unclear setting out.
3	One error with correct procedure	
2	Expanded $(1-\sin^2 x)^n$	
1	Expressing the integral as $\int_0^{\frac{\pi}{2}} \cos x (1-\sin^2 x)^n\, dx$	

(iii) Hence, or otherwise, evaluate $\int_0^{\frac{\pi}{2}} (\cos x)^5\, dx$

$$\int_0^{\frac{\pi}{2}} (\cos x)^5\, dx = \sum_{k=0}^{2} \binom{2}{k}\frac{(-1)^k}{2k+1}$$

note: $2n+1=5 \Rightarrow n=2$

$$= \binom{2}{0}\frac{(-1)^0}{2(0)+1} + \binom{2}{1}\frac{(-1)^1}{2(1)+1} + \binom{2}{2}\frac{(-1)^2}{2(2)+1}$$

$$= 1 + 2\times\frac{-1}{3} + \frac{1}{5}$$

$$= \frac{8}{15}$$

Mark	Guideline	Marker's comment
2	Correct response	Generally, answered well by all students.
1	Correct procedure with an error (e.g. incorrect value of n)	

(c) A vector equation is given by $\underset{\sim}{v} = \underset{\sim}{c} + \underset{\sim}{a}\cos t + \underset{\sim}{b}\sin t$ where $\underset{\sim}{a} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}, \underset{\sim}{b} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$ and $\underset{\sim}{c} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$. **4**

Find the Cartesian equation of the vector and describe it geometrically.

Solution

$$\underset{\sim}{v} = \underset{\sim}{c} + \underset{\sim}{a}\cos t + \underset{\sim}{b}\sin t$$

$$\underset{\sim}{v} - \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}\cos t + \begin{bmatrix} -2 \\ -2 \end{bmatrix}\sin t$$

$$\left\| \begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 3 \\ -2 \end{bmatrix} \right\|^2 = \left| \begin{bmatrix} 2 \\ -2 \end{bmatrix}\cos t + \begin{bmatrix} -2 \\ -2 \end{bmatrix}\sin t \right|^2$$

$$\begin{aligned}(x-3)^2 + (y+2)^2 &= (2\cos t - 2\sin t)^2 + (-2\cos t - 2\sin t)^2 \\ &= 4\cos^2 t - 8\cos t\sin t + 4\sin^2 t + 4\cos^2 t + 8\cos t\sin t + 4\sin^2 t \\ &= 8\left(\sin^2 t + \cos^2 t\right)\end{aligned}$$

$$(x-3)^2 + (y+2)^2 = 8$$

This is the equation of a circle with centre $(3,-2)$ and radius $2\sqrt{2}$

Mark	Guideline	Marker's comment
4	Correct response	Most students knew how to make start but struggled to demonstrate their understanding and solution clearly, making it difficult to achieve full marks. Students who skipped significant lines of reasoning lost marks.
3	Correct equation with no geometrical description	
2	Finding $(x-3)^2 + (y+2)^2 = (2\cos t - 2\sin t)^2 + (-2\cos t - 2\sin t)^2$	
1	Partial solution	

Question 16. (15 marks) Use a separate writing booklet.

(a) Given $I_n = \int_0^1 \frac{1}{\left(1+x^4\right)^n}dx$, show that $I_{n+1} = \frac{4n-1}{4n}I_n + \frac{1}{n2^{n+2}}$

Solution

$$I_n = \int_0^1 \frac{1}{\left(1+x^4\right)^n}dx$$

$$= \left[\frac{x}{\left(1+x^4\right)^n}\right]_0^1 - \int_0^1 x\times(-n)\left(1+x^4\right)^{-n-1}\left(4x^3\right)dx$$

$$= \frac{1}{2^n} + 4n\int_0^1 x^4\left(1+x^4\right)^{-n-1}dx$$

$$= \frac{1}{2^n} + 4n\int_0^1 \left(1+x^4-1\right)\left(1+x^4\right)^{-n-1}dx$$

$$= \frac{1}{2^n} + 4n\int_0^1 \left(1+x^4\right)^{-n} - \left(1+x^4\right)^{-n-1}dx$$

$$= \frac{1}{2^n} + 4n\int_0^1 \frac{1}{\left(1+x^4\right)^n}dx - 4n\int_0^1 \frac{1}{\left(1+x^4\right)^{n+1}}dx$$

$$I_n = \frac{1}{2^n} + 4nI_n - 4nI_{n+1}$$

$$4nI_{n+1} = \frac{1}{2^n} + 4nI_n - I_n$$

$$I_{n+1} = \frac{1}{4n\times 2^n} + \frac{(4n-1)I_n}{4n}$$

$$I_{n+1} = \frac{1}{2^2 n\times 2^n} + \frac{(4n-1)I_n}{4n}$$

$$= \frac{1}{n\times 2^{n+2}} + \frac{(4n-1)I_n}{4n}$$

Mark	Guideline	Marker's comment
3	Correct response	
2	For correct procedure with an error	
1	For finding $I_n = \frac{1}{2^n} + 4n\int_0^1 x^4\left(1+x^4\right)^{-n-1}dx$	

(b) Frank decides to go bungy-jumping. Frank falls, from rest, off a bridge attached by an elastic cable of length *l* metres. After Frank falls *l* metres he is slowed by the cable which exerts a force of $mgk(x-l)$, where *m* is his mass in kilograms, *g* is the constant acceleration due to gravity in ms^{-2}, *k* is the constant of proportionality. Let *x* m be the distance Frank has fallen and let v ms^{-1} be his speed at *x*.

There is negligible air resistance for the duration of Frank's jump.

(i) Show that $v^2 = 2gx - kg(x-l)^2$ for $x > l$ **4**

Solution

$$F = mg - mgk(x-l)$$
$$\ddot{x} = g - gk(x-l)$$
$$\frac{1}{2}\frac{dv^2}{dx} = g - gk(x-l)$$
$$\frac{dv^2}{dx} = 2g - 2gk(x-l)$$
$$v^2 = 2gx - 2gk\frac{(x-l)^2}{2} + C$$

Note: you can't sub in $x = 0,\ v = 0$ to find C as this equation is used when $x > l$ (ie the Bungy rope applies resistance). Prior to that it's free fall. In order to find C we need to find the velocity when $x = l$.

$$\ddot{x} = g$$
$$\frac{d\frac{1}{2}v^2}{dx} = g$$
$$\frac{1}{2}v^2 = gx + D$$

When $x = 0, v = 0 \Rightarrow D = 0$

$$\frac{1}{2}v^2 = gx$$

When $x = l, v^2 = 2gl$

$$2gl = 2g(l) - gk(l-l)^2 + C$$
$$C = 0$$
$$v^2 = 2gx - gk(x-l)^2 \quad \#$$

Mark	Guideline	Marker's comment
4	Correct response	
3	For recognising separate equations	
2	For finding $v^2 = 2gx - 2gk\frac{(x-l)^2}{2} + C$	
1	$\ddot{x} = g - gk(x-l)$	

(ii) Show that Franks fall is first halted at $x = l + \frac{1}{k} + \sqrt{\frac{2l}{k} + \frac{1}{k^2}}$ **2**

Solution

When Frank's fall is halted $v = 0$

$$2gx - gk(x-l)^2 = 0$$
$$2x - k(x^2 - 2lx + l^2) = 0$$
$$kx^2 - 2klx - 2x + kl^2 = 0$$
$$kx^2 - (2kl+2)x + kl^2 = 0$$

$$x = \frac{2kl + 2 \pm \sqrt{(2kl+2)^2 - 4k(kl^2)}}{2k}$$
$$= \frac{2kl + 2 \pm \sqrt{4k^2l^2 + 8kl + 4 - 4k^2l^2}}{2k}$$
$$= \frac{2kl + 2 \pm 2\sqrt{2kl+1}}{2k}$$
$$= \frac{kl + 1 \pm \sqrt{2kl+1}}{k}$$
$$= \frac{kl+1}{k} \pm \frac{1}{k}\sqrt{2kl+1}$$
$$= l + \frac{1}{k} + \sqrt{\frac{2l}{k} + \frac{1}{k^2}}$$

since $l + \frac{1}{k} - \frac{1}{k}\sqrt{2kl+1} < l$ as $\sqrt{2kl+1} > 1$ and it needs to be $> l$

Mark	Guideline	Marker's comment
3	Correct response	
2	For response that does not explain why $x \neq l + \frac{1}{k} - \frac{1}{k}\sqrt{2kl+1}$	
1	For finding	

(iii) Show that if the length of rope is 40 m and the constant of proportionality k is $\frac{1}{10}$ then the bridge needs to be at least 80 metres above the ground. **1**

Solution

$l = 40, k = \frac{1}{10}$

$$x = 40 + \frac{1}{\frac{1}{10}} + \sqrt{\frac{2(40)}{\frac{1}{10}} + \frac{1}{\left(\frac{1}{10}\right)^2}}$$
$$= 80\ m$$

Therefore the bridge needs to be greater than 80 metres.

Mark	Guideline	Marker's comment
1	Correct response	

(c) From a horizontal plane the path of particle projected at an angle α is given by the vector

$$\underset{\sim}{r}(t) = Vt\cos\alpha\,\underset{\sim}{i} + \left(Vt\sin\alpha - \frac{gt^2}{2}\right)\underset{\sim}{j}$$

Where V in ms^{-1} is the initial projection speed, *t* is the time in seconds and *g* is the acceleration due to gravity.

(i) Show that the Cartesian path of the particle is given by $y = x\tan\alpha - \dfrac{gx^2\sec^2\alpha}{2V^2}$ **2**

Solution

$x = Vt\cos\alpha$

$t = \dfrac{x}{V\cos\alpha}$

$$\begin{aligned} y &= Vt\sin\alpha - \frac{1}{2}gt^2 \\ &= V\left(\frac{x}{V\cos\alpha}\right)\sin\alpha - \frac{1}{2}g\left(\frac{x}{V\cos\alpha}\right)^2 \\ &= \frac{x}{\cos\alpha}\sin\alpha - \frac{1}{2}g\frac{x^2}{V^2\cos^2\alpha} \\ &= x\tan\alpha - \frac{gx^2\sec^2\alpha}{2V^2} \end{aligned}$$

Mark	Guideline	Marker's comment
2	Correct response	
1	Isolating the parameter *t*	

(ii) A vertical wall is a distance *d* metres from the origin.

Show that if $d < \dfrac{V^2}{g}$ the particle will strike the wall provided that $\beta < \alpha < \dfrac{\pi}{2} - \beta$, where

$\beta = \dfrac{1}{2}\sin^{-1}\left(\dfrac{gd}{V^2}\right)$ **3**

Solution

The range of the projectile is found when $y = 0$

$$x\tan\alpha - \frac{gx^2\sec^2\alpha}{2V^2} = 0$$

$$x\tan\alpha\left(1 - \frac{gx}{2\sin\alpha\cos\alpha V^2}\right) = 0$$

$$x = \frac{V^2\sin 2\alpha}{g} \quad as \quad x \neq 0$$

Let β be the angle of projection whose range is *d*.

Another angle that has a range of *d* is the complement of β .

Consider the complementary angle of $\beta = \frac{\pi}{2} - \beta$

$$
\begin{aligned}
x &= \frac{V^2 \sin\left[2\left(\frac{\pi}{2} - \beta\right)\right]}{g} \\
&= \frac{V^2 \sin(\pi - 2\beta)}{g} \\
&= \frac{V^2 \sin 2\beta}{g} \\
&= d
\end{aligned}
$$

Therefore an angle of projection α that is between β and its complement will have a greater range and hit the wall.

This $\quad \beta < \alpha < \frac{\pi}{2} - \beta$

Now the range d is found by

$$
\begin{aligned}
d &= \frac{V^2 \sin 2\beta}{g} \\
\sin 2\beta &= \frac{dg}{V^2} \\
2\beta &= \sin^{-1}\left(\frac{dg}{V^2}\right) \\
\beta &= \frac{1}{2}\sin^{-1}\left(\frac{dg}{V^2}\right)
\end{aligned}
$$

Therefore $\beta < \alpha < \frac{\pi}{2} - \beta \qquad$ where $\beta = \frac{1}{2}\sin^{-1}\left(\frac{dg}{V^2}\right)$

Mark	Guideline	Marker's comment
2	Correct response	
1	Partial solution	